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The exact static exterior and interior metric of a thick plane plate

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Abstract. The exact static exterior and interior solution of Einstein's equations for a plane thick disk is obtained. It has been shown that the corresponding energy–momentum tensor fulfils the generalized O'Brien–Synge junction conditions. The correspondence of this tensor to the surface energy–momentum tensor of a thin plane plate is demonstrated. Our disk being neither very thick nor very dense, we have also obtained a connection with the Newtonian theory.

1. Introduction

Kuchowicz (1968) asserted that only a few exact solutions of the gravitational field equations for space filled with matter exist. In spite of many published papers concerning the spherical and cylindrical symmetries in general relativity (e.g. Harrison *et al.* 1965, Bonnor 1965, Banerji 1968, Langer 1968) one cannot conclude that plane symmetry is less interesting. Its detailed study, however, has not been so intensive.

Some conclusions about thin plane plates (plane shells) were made by Horský (1968). The author employed some general conclusions of Israel and Kuchař (Israel 1966, Kuchař 1968) and showed that the metric of a thin plane shell has the form

$$g_{\alpha\beta}{}^{\pm} = \begin{pmatrix} 1 & & & 0 \\ & (\pm bx+C)^{4/3} & & \\ & & (\pm bx+C)^{4/3} & \\ 0 & & & -(\pm bx+C)^{-2/3} \end{pmatrix}, \qquad \begin{array}{l} b = -3\pi\sigma \\ C = 1 \end{array}$$

under the assumptions that it is made of a perfect fluid at rest and that the shell is situated in the coordinate plane $x^1 = 0$. The surface energy–momentum tensor is

$$t_a{}^b = \begin{pmatrix} p & & 0 \\ & p & \\ 0 & & -\sigma \end{pmatrix}$$

and it was shown that there must be a very strong tension

$$p = -\tfrac{1}{4}\sigma \tag{1}$$

in our shell.

The notation is that of Israel (1966). It should be noted that the Greek subscripts assume the values 1–4 and the Latin ones 2–4. The signature of the fundamental quadratic form is $(1, 1, 1, -1)$, $x^4 = t$, $c = G = 1$, $\kappa = 8\pi$; the Einstein equations are of the same form as they are in Møller's (1961) textbook.

2. The finite number of the plane shells

By a convenient choice of the origin on the x axis we can achieve an arbitrary value of the constant C. If we measure the x coordinate from the singularity surface we have $C = 0$ and

$$g_{\alpha\beta}{}^{\pm} = \begin{pmatrix} 1 & & & 0 \\ & (bx)^{4/3} & & \\ & & (bx)^{4/3} & \\ 0 & & & -(bx)^{-2/3} \end{pmatrix}.$$

As in the previous notation we let x^+ represent the limiting value of the x coordinate as we approach the shell from the right (the meaning of x^- is obvious from this). The constant b may differ on each side of the shell (b^+, b^-) in the general case. This can be easily seen if there is more than one shell in space.

In our system of coordinates

$$^4g_{11} = 1, \, ^4g_{i1} = 0$$

and the extrinsic curvature tensor of the shell is given by

$$K_{ik}{}^{\pm} = \frac{1}{2}\frac{\partial^4 g_{ik}}{\partial x}\bigg|^{\pm}. \tag{2}$$

This tensor 'jumps' when passing through the shell and it is necessary that

$$\gamma_{ik} = K_{ik}{}^{+} - K_{ik}{}^{-} \equiv [K_{ik}] = -8\pi(t_{ik} - \tfrac{1}{2}g_{ik}t_a{}^a). \tag{3}$$

It follows from (3) that

$$\gamma_2{}^2 = \gamma_3{}^3 = -\tfrac{1}{2}\kappa\sigma, \qquad \gamma_4{}^4 = \tfrac{1}{2}\kappa(2p+\sigma) \tag{4}$$

while from (2) one can obtain

$$\gamma_2{}^2 = \gamma_3{}^3 = -\frac{2}{3x^-} + \frac{2}{3x^+}, \qquad \gamma_4{}^4 = \frac{1}{3x^-} - \frac{1}{3x^+}. \tag{5}$$

If we compare (4) and (5) we have

$$\left.\begin{aligned} -\frac{2}{3x^-} + \frac{2}{3x^+} &= -\frac{\kappa\sigma}{2} \\ -\frac{1}{3x^-} + \frac{1}{3x^+} &= -\frac{\kappa}{2}(2p+\sigma) \end{aligned}\right\}. \tag{6}$$

If we impose the condition that these equations are valid simultaneously, we can conclude that

$$p = -\frac{\sigma}{4}$$

which is in agreement with (1). Let us now consider a static system of n thin plane shells (n denotes a natural number) that is symmetric with respect to the shell denoted by O. Let the distance δ between any two of the shells be constant. It is convenient to introduce the notation

$$x_0 \equiv x_0{}^+, x_1 \equiv x_1{}^+, \ldots, x_m \equiv x_m{}^+$$

and similarly

$$b_0 \equiv b_0{}^+, b_1 \equiv b_1{}^+, \ldots, b_m \equiv b_m{}^+.$$

The gravitational field is subject to the conditions

$$[g_{ab}] = 0$$

and to those given by (3). From them we obtain

$$\left.\begin{aligned} \frac{2}{3x_m{}^-} - \frac{2}{3x_m{}^+} = \frac{\kappa\sigma_m}{2}, &\qquad p_m = -\tfrac{1}{4}\sigma_m \\ b_m{}^-|x_m{}^-| = b_m{}^+&|x_m{}^+| \end{aligned}\right\}. \tag{7}$$

Because of the symmetry of the system ($x_0{}^- = -x_0{}^+$) we have

$$x_0 \equiv x_0{}^+ = \frac{-8}{3\kappa\sigma}. \tag{8}$$

and from (7) one can obtain the recurrence formulae

$$x_m = \frac{x_{m-1}+\delta}{1-\frac{3}{4}\kappa\sigma_m(x_{m-1}+\delta)} \tag{9}$$

$$b_m = \{-\tfrac{3}{4}\kappa\sigma_m(x_{m-1}+\delta)+1\}b_{m-1}. \tag{10}$$

3. The transition to a thick and static plane plate

Let us take the limit for $\delta \to 0$ in (9) and (10) in such a way that $\sigma/\delta \to \epsilon(x)$, where $\epsilon(x)$ is a given function (a space density of matter in the plane disk).

Let us calculate from first principles the derivatives

$$\frac{\mathrm{d}a}{\mathrm{d}x} \equiv \lim_{\substack{\delta\to 0\\ \sigma/\delta\to\epsilon(x)}} \frac{x_m - x_{m-1}}{\delta} = \lim_{\substack{\delta\to 0\\ \sigma/\delta\to\epsilon(x)}} \left[\left\{\frac{4(x_{m-1}+\delta)}{4-3\kappa\sigma_m(x_{m-1}+\delta)} - x_{m-1}\right\}\delta^{-1}\right]$$

$$= 1+\tfrac{3}{4}\kappa\epsilon(x)a^2(x).$$

and

$$\left.\begin{aligned}\frac{\mathrm{d}b}{\mathrm{d}x} \equiv \lim_{\substack{\delta\to 0\\ \sigma/\delta\to\epsilon(x)}} \frac{b_m - b_{m-1}}{\delta} &= \lim_{\substack{\delta\to 0\\ \sigma/\delta\to\epsilon(x)}} \frac{b_{m-1}\{-\frac{3}{4}\kappa\sigma_m(x_{m-1}+\delta)\}}{\delta}\\ &= -\tfrac{3}{4}\kappa\epsilon(x)b(x)a(x).\end{aligned}\right\}. \tag{11}$$

The solution of (11) is

$$b = C\exp\left\{-\tfrac{3}{4}\kappa\int\epsilon(x)a(x)\,\mathrm{d}x\right\} \tag{12}$$

where $a(x)$ is given by the equation

$$\frac{\mathrm{d}a}{\mathrm{d}x} = 1+\tfrac{3}{4}\kappa\epsilon(x)a^2(x) \tag{13}$$

under the condition

$$\lim_{x\to 0^+} a(x) = -\infty$$

following from (8).

Let the thickness of the disk measured along the x axis be $2L$. The interior solution can also be written in the form

$$g_{\alpha\beta}(x) = \begin{pmatrix}1 & & & 0\\ & \{b(x)a(x)\}^{4/3} & & \\ & & \{b(x)a(x)\}^{4/3} & \\ 0 & & & -\{b(x)a(x)\}^{-2/3}\end{pmatrix} \qquad |x| \leqslant L$$

and for the exterior solution we have

$$g_{\alpha\beta}(x) = \begin{pmatrix}1 & & & 0\\ & [F(L,x)]^{4/3} & & \\ & & [F(L,x)]^{4/3} & \\ 0 & & & -[F(L,x)]^{-2/3}\end{pmatrix} \qquad |x| \geqslant L$$

$$F(L,x) \equiv b(L)\{a(L)+x-L\}.$$

The explicit forms of this solution can be easily obtained for ϵ = constant (homogeneous plane disk). From (13) it follows that

$$\int_0^x \mathrm{d}x = \int_{-\infty}^{a} \frac{\mathrm{d}a}{1+\frac{3}{4}\kappa\epsilon a^2}$$

so that

$$a(x) = -\frac{2}{(3\kappa\epsilon)^{1/2}}\cot\{\tfrac{1}{2}(3\kappa\epsilon)^{1/2}x\}. \tag{14}$$

Equation (12) gives

$$b(x) = K' \sin\{\tfrac{1}{2}(3\kappa\epsilon)^{1/2}x\}$$

and the interior solution is

$$g_{\alpha\beta}(x) = \begin{pmatrix} 1 & & & 0 \\ & [G(x)]^{4/3} & & \\ & & [G(x)]^{4/3} & \\ 0 & & & -[G(x)]^{-2/3} \end{pmatrix} \qquad |x| \leqslant L$$

$$G(x) = \frac{2K'}{(3\kappa\epsilon)^{1/2}}\cos\{\tfrac{1}{2}(3\kappa\epsilon)^{1/2}x\}.$$

A constant K' is not yet determined. After a simple transformation

$$x' = x, y' = \left\{\frac{2K'}{(3\kappa\epsilon)^{1/2}}\right\}^{2/3} y, \qquad z' = \left\{\frac{2K'}{(3\kappa\epsilon)^{1/2}}\right\}^{2/3} z, \qquad t' = \left\{\frac{2K'}{(3\kappa\epsilon)^{1/2}}\right\}^{-1/3} t$$

one can write the interior solution in the form

$$g_{\alpha\beta}(x) = \begin{pmatrix} 1 & & & 0 \\ & +[\cos\{\frac{1}{2}(3\kappa\epsilon)^{1/2}x\}]^{4/3} & & \\ & & +[\cos\{\frac{1}{2}(3\kappa\epsilon)^{1/2}x\}]^{4/3} & \\ 0 & & & -[\cos\{\frac{1}{2}(3\kappa\epsilon)^{1/2}x\}]^{-2/3} \end{pmatrix} \qquad |x| \leqslant L. \tag{15}$$

4. The energy–momentum tensor

Following Synge's 'g-method' (Synge 1960) let us calculate the energy–momentum tensor of a thick and homogeneous plane disk.

From (15) it follows that the non-zero $\Gamma^{\alpha}_{\beta\gamma}$ are

$$\Gamma^1_{22} = \Gamma^1_{33} = -\tfrac{1}{2}g_{22,1} = (\tfrac{1}{3}\kappa\epsilon)^{1/2}[\cos\{\tfrac{1}{2}(3\kappa\epsilon)^{1/2}x\}]^{1/3}\sin\{\tfrac{1}{2}(3\kappa\epsilon)^{1/2}x\}$$

$$\Gamma^1_{44} = -\tfrac{1}{2}g_{44,1} = \tfrac{1}{2}(\tfrac{1}{3}\kappa\epsilon)^{1/2}[\cos\{\tfrac{1}{2}(3\kappa\epsilon)^{1/2}x\}]^{-5/3}\sin\{\tfrac{1}{2}(3\kappa\epsilon)^{1/2}x\}$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \tfrac{1}{2}g^{33}g_{33,1} = -(\tfrac{1}{3}\kappa\epsilon)^{1/2}\tan\{\tfrac{1}{2}(3\kappa\epsilon)^{1/2}x\}$$

$$\Gamma^4_{14} = \tfrac{1}{2}g^{44}g_{44,1} = \tfrac{1}{2}(\tfrac{1}{3}\kappa\epsilon)^{1/2}\tan\{\tfrac{1}{2}(3\kappa\epsilon)^{1/2}x\}.$$

For the mixed Ricci tensor $R_{\alpha}{}^{\beta}$ we have

$$R_{\alpha}{}^{\beta} \equiv g^{\delta\beta}R_{\delta\alpha} = \begin{pmatrix} -\frac{3}{4}\kappa\epsilon & & & 0 \\ & -\frac{1}{2}\kappa\epsilon & & \\ & & -\frac{1}{2}\kappa\epsilon & \\ 0 & & & \frac{1}{4}\kappa\epsilon \end{pmatrix}$$

so that

$$R \equiv R_{\alpha}{}^{\alpha} = -\tfrac{3}{2}\kappa\epsilon.$$

Immediately from the Einstein equations ($G_{\alpha\beta} = -\kappa T_{\alpha\beta}$) we obtain

$$T_\alpha{}^\beta = -\frac{R_\alpha{}^\beta}{\kappa} - \tfrac{3}{4}\delta_\alpha{}^\beta\epsilon = \begin{pmatrix} 0 & & & 0 \\ & -\frac{1}{4}\epsilon & & \\ & & -\frac{1}{4}\epsilon & \\ 0 & & & -\epsilon \end{pmatrix}.$$

It can be easily proved that the generalized O'Brien–Synge junction conditions (Israel 1966)

$$G_{\alpha\beta}m^\alpha m^\beta|^+ = G_{\alpha\beta}m^\alpha m^\beta|^-$$
$$G_{\alpha\beta}l_{(a)}{}^\alpha m^\beta|^+ = G_{\alpha\beta}l_{(a)}{}^\alpha m^\beta|^-$$

are satisfied. It is necessary only to take into consideration that $\psi|^+ = 0$, $T_1{}^1 = 0$, $m_\alpha = \delta_\alpha{}^1$ and $l_{(a)}{}^\alpha = \delta_a{}^\alpha$ at the point $x = L$.

5. Remarks

(*a*) Some objections to the interpretation of our $T_{\alpha\beta}$ may exist. To remove these let us perform 'the inverse' limiting transition to the surface energy–momentum tensor t_{ik}. For the t_{44} component we have

$$t_{44} = \lim_{\delta\to 0}\int_{x_0}^{x_0+\delta} T_{44}\,\mathrm{d}x = \lim_{\delta\to 0}\int_{x_0}^{x_0+\delta}\frac{\sigma}{\delta}(\cos\alpha x)^{-2/3}\,\mathrm{d}x$$
$$= \lim_{\delta\to 0}\frac{\sigma}{\delta}(\cos\alpha x_0)^{-2/3}\delta = \sigma(\cos\alpha x_0)^{-2/3}$$

so that

$$t_4{}^4 = g^{44}t_{44} = -\sigma$$

as was expected. For the other components of t_{ik} the calculations are similar.

(*b*) The essential singularity of the exterior solution is at the point

$$x = L - a(L).$$

Indeed, for our singularity to lie outside the disk it must be

$$a(L) < 0.$$

If the relation

$$\tfrac{1}{2}(3\kappa\epsilon)^{1/2}L < \tfrac{1}{2}\pi$$

holds, there is a singularity outside the disk. For

$$\tfrac{1}{2}(3\kappa\epsilon)^{1/2}L = \tfrac{1}{2}\pi$$

the singularity is shifted to the surface of the disk. If the argument increases again ($\tfrac{1}{2}(3\kappa\epsilon)^{1/2}L > \tfrac{1}{2}\pi$), the essential singularity will appear in the interior solution too.

(*c*) Let us compare our results with those in the Newtonian theory. We shall calculate the acceleration of a free probe particle immediately above the disk. If this particle starts with zero velocity it will move along the straight line y = const., z = const. At this moment its acceleration (measured by a standard clock at rest at this point) is given by

$$\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} \equiv g_{\mathrm{rel}} = \frac{1}{3a(L)}.$$

Using (14) we obtain

$$g_{\mathrm{rel}} = -\tfrac{1}{6}(3\kappa\epsilon)^{1/2}\tan\{\tfrac{1}{2}(3\kappa\epsilon)^{1/2}L\}.$$

Let $\tfrac{1}{2}(3\kappa\epsilon)^{1/2}L \ll 1$, so that $\tan\{\tfrac{1}{2}(3\kappa\epsilon)^{1/2}L\} \simeq \tfrac{1}{2}(3\kappa\epsilon)^{1/2}L$ and

$$g_{\mathrm{rel}} = -\tfrac{1}{4}\kappa\epsilon L = -2\pi\epsilon L.$$

Let the homogeneous plane disk (of thickness $2L$) have the same density in the Newtonian theory. By direct use of the Gauss theorem it can be easily shown that the acceleration of the probe particle is given by the expression

$$g_{\text{clas}} = -4\pi\epsilon L.$$

The two values for g are not the same. This result can be explained in the following way: according to Landau and Lifshitz (1967) the total energy of matter and the static gravitational field is

$$-P_4 = M = \int (T_1{}^1 + T_2{}^2 + T_3{}^3 - T_4{}^4)(-g)^{1/2}\,\mathrm{d}V.$$

Let us calculate the total energy of a (abstract) piece of our disk. Because $(-g)^{1/2} \simeq 1$, we obtain

$$M \simeq \int (T_2{}^2 + T_3{}^3 - T_4{}^4)\,\mathrm{d}V = \tfrac{1}{2}\epsilon \int \mathrm{d}V = \tfrac{1}{2}\epsilon V.$$

The Newtonian density of matter is the relation between the total mass and its volume. We have shown that this factor is $\frac{1}{2}\epsilon$ (and not ϵ). We obtain, in this case,

$$g_{\text{clas}} = -2\pi\epsilon L$$

and so

$$g_{\text{clas}} = g_{\text{rel}}.$$

This very interesting correspondence holds in our approximation only.

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